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ORIGINAL RESEARCH

Dynamics in a periodic two-species predator–prey system with pure delays

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Abstract A class of non-autonomous two-species Lotka–Volterra predator–prey system with pure discrete time delays is discussed. Some sufficient conditions on the boundedness, permanence, extinction, positive periodic solution and global attractivity of the system are established by means of the comparison method, coincidence degree theory and Liapunov functional.

Keywords Lotka–Volterra predator–prey system · Discrete time delay · Liapunov functional · Global attractivity · Positive periodic solution.

Introduction

In the real world, there are many types of interactions between two species. Predator–prey relations are among the most common ecological interactions. Remarkably, the whole field of mathematical ecology began with the studies of population dynamics subject to the predator–prey interaction, that is, with the classical works by Lotka [1] and Volterra [2]. Traditional two-species non-autonomous Lotka–Volterra predator–prey systems take the form

$$\begin{aligned}\frac{dx_1(t)}{dt} &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t)\right],\\ \frac{dx_2(t)}{dt} &= x_2(t)\left[-r_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t)\right].\end{aligned} \tag{1}$$

Recently, the properties of system (1) are discussed by many scholars [3–7]. Some sufficient conditions are obtained for the persistence, permanence and extinction of the species, the existence and uniqueness of periodic solutions or almost periodic solutions, and the global stability of solutions for system (1).

However, in the real world, the growth rate of a natural species will not often respond immediately to changes in its own population or that of an interacting species, but will rather do so after a time lag [8]. Time delays have a great destabilizing influence on the species population; this result was put forwarded by May [9].

Therefore, we should introduce time delay into model foundation, which will have more resemblance to the real ecosystem.

In this paper, we investigate the following two-species Lotka–Volterra type predator–prey systems with pure discrete time delays

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1(t) - a_{11}(t)x_1(t-\tau_1) - a_{12}(t)x_2(t-\tau_1)\right],\\ \dot{x}_2(t) &= x_2(t)\left[-r_2(t) + a_{21}(t)x_1(t-\tau_2) - a_{22}(t)x_2(t-\tau_2)\right].\end{aligned} \tag{2}$$

Our main purpose is to establish some sufficient conditions on the boundedness, permanence, extinction, positive periodic solution and global attractivity of the system (2).

The organization of this paper is as follows. In the next section, we will present some basic assumptions and main lemmas. In "Main results", we will consider conditions for the boundedness, permanence, extinction, positive periodic solution and global attractivity of the system. In the final section, as an application, one special case of the system is considered.

Preliminaries

In system (2), we have that $x_1(t)$ is the prey population density and $x_2(t)$ is the predator population density, $r_1(t)$ and

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$a_{11}(t)$ are the intrinsic growth rate and density-dependent coefficient of the prey, respectively, $r_2(t)$ is the intrinsic growth rate of the predator, $a_{12}(t)$ is the capturing rate of the predator and $a_{21}(t)$ is the rate of conversion of nutrients into the reproduction of the predator. Throughout this paper, for system (2) we introduce the following hypotheses.

(H_1) $\tau_i > 0 (i = 1, 2)$ are positive constants, $r_i(t) (i = 1, 2)$ are continuous ω-periodic functions with $\int_0^{\omega} r_i(t)dt > 0$. $a_{ij}(t)(i, j = 1, 2)$ are continuous positive ω-periodic functions.

From the viewpoint of mathematical biology, in this paper for system (2), we only consider the solution with the following initial condition:

$$x_i(t) = \phi_i(t) \quad \text{for all} \quad t \in [-\tau, 0), \quad i = 1, 2, \tag{3}$$

where $\phi_i(t)$ $(i = 1, 2)$ are nonnegative continuous functions defined on $[-\tau, 0)$ satisfying $\phi_i(0) > 0$ $(i = 1, 2)$ and $\tau = \max\{\tau_1, \tau_2\}$.

In this paper, for any ω-periodic continuous function $f(t)$, we denote

$$f^L = \min_{t \in [0,\omega]} f(t), \quad f^M = \max_{t \in [0,\omega]} f(t), \quad \bar{f} = \frac{1}{\omega} \int_0^{\omega} f(t)dt.$$

To obtain the existence of positive ω-periodic solutions of system (2), we will use the continuation theorem. For the reader's convenience, we will introduce the continuation theorem in the following. First we define some definitions.

Let X and Z be two normed vector spaces. Let $L :$ Dom $L \subset X \to Z$ be a linear operator and $N : X \to Z$ be a continuous operator. The operator L is called a Fredholm operator of index zero, if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is a closed set in Z. If L is a Fredholm operator of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_P and denoted by $J : ImQ \to KerL$ an isomorphism of ImQ onto $KerL$. Let Ω be a bounded open subset of X, we say that the operator N is L-compact on $\overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω in X, if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Such definitions can be found in [10–12].

Now, we present some useful lemmas.

Lemma 1 Set $R_+^2 = \{(x_1, x_2) : x_i > 0, i = 1, 2\}$ is positively invariant for system (2).

The proof of Lemma 1 is simple, and here we omit it.

Lemma 2 [(see,[13])] Consider the following equation: $\dot{u}(t) = u(t)(d_1 - d_2u(t))$, where $d_2 > 0$, we have If $d_1 > 0$, *then* $\lim_{t\to+\infty} u(t) = d_1/d_2$. If $d_1 < 0$, *then* $\lim_{t\to+\infty} u(t) = 0$.

Lemma 3 (see [4]) Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. If

(a) for each $\lambda \in (0, 1)$ and $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;
(b) for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;
(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,

then the operator equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Main results

In this section, we will obtain some sufficient conditions for the boundedness, existence of periodic solution, global attractivity, permanence, and extinction of system (2).

Theorem 1 Suppose that assumption (**H1**) holds, then there exist positive constants $M_i (i = 1, 2)$ such that

$$x_i(t) \leq M_i,$$

for any positive solution $x_i(t)$ of system (2).

Proof Let $(x_1(t), x_2(t))$ be a solution of system (2). Firstly, it follows from the first equation of system (2) that for $t > \tau$, we have

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)[r_1(t) - a_{11}(t)x_1(t - \tau_1) - a_{12}(t)x_2(t - \tau_1)] \\ &\leq x_1(t)[r_1^M - a_{11}^L e^{-r_1^M \tau_1} x_1(t)] \quad \text{for} \quad t > \tau. \end{aligned}$$

We consider the following auxiliary equation

$$\frac{du(t)}{dt} = u(t)[r_1^M - a_{11}^L e^{-r_1^M \tau_1} u(t)].$$

By Lemma 2, we derive

$$\lim_{t\to+\infty} u(t) = \frac{r_1^M e^{r_1^M \tau_1}}{a_{11}^L} =: M_1.$$

By comparison, there exists a $T_1 > \tau$ such that $x_1(t) \leq M_1$ for $t \geq T_1$.

Next, from the second equation of system (2) for $t > T_1$, we have

$$\begin{aligned} \frac{dx_2(t)}{dt} &\leq x_2(t)[a_{21}^M M_1 - a_{22}^L x_2(t - \tau_2)] \\ &\leq x_2(t)[a_{21}^M M_1 - a_{22}^L e^{-a_{21}^M M_1 \tau_2} x_2(t)] \quad \text{for} \quad t > T_1. \end{aligned}$$

We consider the following auxiliary equation

$$\frac{du(t)}{dt} = u(t)[a_{21}^M M_1 - a_{22}^L e^{-a_{21}^M M_1 \tau_2} u(t)].$$

By Lemma 2, we derive

$$\lim_{t\to+\infty} u(t) = \frac{a_{21}^M M_1 e^{a_{21}^M M_1 \tau_2}}{a_{22}^L} =: M_2.$$

By comparison, there exists a $T_2 > \tau$ such that $x_2(t) \le M_2$ for $t \ge T_2$. This completes the proof. $\square$

Theorem 2 Suppose that assumption (**H1**) holds and $\bar{r}_1 - (\frac{a_{11}}{a_{21}})^M \bar{r}_2 > 0$. Then system (2) has at least one positive $\omega-$ periodic solution.

Proof Let

$$x_1(t) = \exp\{u_1(t)\} \quad \text{and} \quad x_2(t) = \exp\{u_2(t)\}.$$

Then system (2) is rewritten in the following system

$$\begin{aligned}\dot{u}_1(t) &= r_1(t) - a_{11}(t)\exp\{u_1(t-\tau_1)\} - a_{12}(t)\exp\{u_2(t-\tau_1)\},\\ \dot{u}_2(t) &= -r_2(t) + a_{21}(t)\exp\{u_1(t-\tau_2)\} - a_{22}(t)\exp\{u_2(t-\tau_2)\}.\end{aligned} \tag{4}$$

To apply Lemma 3 to system (4), we introduce the normed vector spaces X and Z as follows. Let $C(R, R^2)$ denote the space of all continuous functions $u(t) = (u_1(t), u_2(t)) : R \to R^2$. We take

$$X = Z = \{u(t) \in C(R, R^2) : u(t) \ \text{ an } \omega\text{−periodic function}\}$$

with norm

$$\| u \| = \max_{t\in[0,\omega]} |u_1(t)| + \max_{t\in[0,\omega]} |u_2(t)|.$$

It is obvious that X and Z are the Banach spaces. We define a linear operator $L : \operatorname{Dom} L \subset X \to Z$ and a continuous operator $N : X \to Z$ as follows.

$$Lu(t) = \dot{u}(t)$$

and

$$Nu(t) = (Nu_1(t), Nu_2(t)). \tag{5}$$

where

$$\begin{aligned}Nu_1(t) &= r_1(t) - a_{11}(t)\exp\{u_1(t-\tau_1)\} - a_{12}(t)\exp\{u_2(t-\tau_1)\},\\ Nu_2(t) &= -r_2(t) + a_{21}(t)\exp\{u_1(t-\tau_2)\} - a_{22}(t)\exp\{u_2(t-\tau_2)\}.\end{aligned}$$

Further, we define continuous projectors $P : X \to X$ and $Q : Z \to Z$ as follows.

$$Pu(t) = \frac{1}{\omega}\int_0^{\omega} u(t)dt, \quad Qv(t) = \frac{1}{\omega}\int_0^{\omega} v(t)dt.$$

We easily see $\operatorname{Im} L = \{v \in Z : \int_0^{\omega} v(t)dt = 0\}$ and $\operatorname{Ker} L = R^2$. It is obvious that $\operatorname{Im} L$ is closed in Z and $\dim\operatorname{Ker} L = 2$. Since for any $v \in Z$ there are unique $v_1 \in R^n$ and $v_2 \in \operatorname{Im} L$ with

$$v_1 = \frac{1}{\omega}\int_0^{\omega} v(t)dt, \quad v_2(t) = v(t) - v_1,$$

such that $v(t) = v_1 + v_2(t)$, we have $\operatorname{codim}\operatorname{Im} L = 2$. Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given in the following form:

$$K_p v(t) = \int_0^t v(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t v(s)dsdt.$$

For convenience, we denote $F(t) = (F_1(t), F_2(t))$ as follows

$$\begin{aligned}F_1(t) &= r_1(t) - a_{11}(t)\exp\{u_1(t-\tau_1)\} - a_{12}(t)\exp\{u_2(t-\tau_1)\},\\ F_2(t) &= -r_2(t) + a_{21}(t)\exp\{u_1(t-\tau_2)\} - a_{22}(t)\exp\{u_2(t-\tau_2)\}.\end{aligned} \tag{6}$$

Thus, we have

$$QNu(t) = \frac{1}{\omega}\int_0^{\omega} F(t)dt \tag{7}$$

and

$$\begin{aligned}K_p(I-Q)Nu(t) &= K_pINu(t) - K_pQNu(t)\\ &= \int_0^t F(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t F(s)dsdt\\ &\quad + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^{\omega} F(s)ds.\end{aligned} \tag{8}$$

From formulas (7) and (8), we easily see that QN and $K_p(I-Q)N$ are continuous operators. Furthermore, it can be verified that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open-bounded set $\Omega \subset X$ by using Arzela–Ascoli theorem and $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ for any open-bounded subset $\Omega \subset X$.

Now, we reach the position to search for an appropriate open-bounded subset Ω for the application of the continuation theorem (Lemma 3) to system (2).

Corresponding to the operator equation $Lu(t) = \lambda Nu(t)$ with parameter $\lambda \in (0, 1)$, we have

$$\dot{u}_i(t) = \lambda F_i(t), \quad i = 1, 2, \tag{9}$$

where $F_i(t)$ $(i = 1, 2)$ are given in Eq. 6.

Assume that $u(t) = (u_1(t), u_2(t)) \in X$ is a solution of system (9) for some parameter $\lambda \in (0, 1)$. By integrating system (9) over the interval $[0, \omega]$, we obtain

$$\begin{aligned}&\int_0^{\omega}\left[r_1(t) - a_{11}(t)\exp\{u_1(t-\tau_1)\} - a_{12}(t)\exp\{u_2(t-\tau_1)\}\right]dt = 0,\\ &\int_0^{\omega}\left[-r_2(t) + a_{21}(t)\exp\{u_1(t-\tau_2)\} - a_{22}(t)\exp\{u_2(t-\tau_2)\}\right]dt = 0.\end{aligned} \tag{10}$$

By (10) we get ,

$$\begin{aligned}&\int_0^{\omega}\left[a_{11}(t)\exp\{u_1(t-\tau_1)\} + a_{12}(t)\exp\{u_2(t-\tau_1)\}\right]dt = \bar{r}_1\omega,\\ &\int_0^{\omega}\left[a_{21}(t)\exp\{u_1(t-\tau_2)\} - a_{22}(t)\exp\{u_2(t-\tau_2)\}\right]dt = \bar{r}_2\omega.\end{aligned} \tag{11}$$

For each $i, j = 1, 2$, we have

$$\begin{aligned}
&\int_0^{\omega} a_{ij}(t)\exp\{u_i(t-\tau_i)\}dsdt \\
&= \int_{-\tau_i}^{\omega-\tau_i} a_{ij}(s+\tau_i)\exp\{u_i(s)\}ds \\
&= \int_0^{\omega} a_{ij}(s+\tau_i)\exp\{u_i(s)\}ds \\
&= \int_0^{\omega} a_{ij}(t+\tau_i)\exp\{u_i(t)\}dt.
\end{aligned} \tag{12}$$

From the continuity of $u(t) = (u_1(t), u_2(t))$, there exist constants $\xi_i, \eta_i \in [0, \omega]$ $(i = 1, 2)$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), \quad i = 1, 2. \tag{13}$$

From (11)–(13) and the condition of Theorem 2, we further obtain

$$\begin{aligned}
&u_i(\eta_i) \le \ln\left(\frac{\bar{r}_1}{\bar{A}_i}\right), \quad u_1(\xi_1) \ge \ln\left(\frac{\bar{r}_2}{\bar{A}_3}\right) =: B_1, \\
&\quad u_2(\xi_2) \ge \ln A_4 =: B_2,
\end{aligned} \tag{14}$$

where

$$A_i(t) = a_{1i}(t+\tau_1),\ A_3(t) = a_{22}(t+\tau_2),$$
$$A_4 = \frac{\bar{r}_1 - (\frac{a_{11}}{a_{21}})^M \bar{r}_2}{\bar{A}_2 + (\frac{a_{11}}{a_{21}})^M \bar{A}_3}, \quad i = 1, 2.$$

From (4) and (11) we have

$$\begin{aligned}
\int_0^{\omega} |\dot{u}_1(t)|dt &\le \int_0^{\omega} \big[r_1(t) + a_{11}(t)\exp\{u_1(t-\tau_1)\} \\
&\quad + a_{12}(t)\exp\{u_2(t-\tau_1)\}\big]dt \\
&\le 2\bar{r}_1\omega =: C_1,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\int_0^{\omega} |\dot{u}_2(t)|dt &\le \int_0^{\omega} \big[r_2(t) + a_{21}(t)\exp\{u_1(t-\tau_2)\} \\
&\quad + a_{22}(t)\exp\{u_2(t-\tau_2)\}\big]dt \\
&\le \frac{2a_{21}^M \bar{r}_1\omega}{a_{11}^L} =: C_2.
\end{aligned} \tag{16}$$

By (14)–(16), we have

$$\begin{aligned}
u_i(t) &\le u_i(\eta_i) + \int_0^{\omega} |\dot{u}_i(t)|dt \le \ln\left(\frac{\bar{r}_1}{\bar{A}_i}\right) \\
&\quad + C_i =: M_i \quad i = 1, 2,
\end{aligned} \tag{17}$$

and

$$u_i(t) \ge u_i(\xi_i) - \int_0^{\omega} |\dot{u}_i(t)|dt \ge B_i - C_i =: N_i \quad i = 1, 2. \tag{18}$$

Therefore, from (17), (18) we have

$$\max_{t\in[0,\ \omega]} |u_i(t)| \le \max\left\{|M_i|, |N_i|\right\} =: H_i, \qquad i = 1, 2.$$

It can be seen that the constants $H_i\,(i = 1, 2)$ are independent of parameter $\lambda \in (0, 1)$.

For any $u = (u_1, u_2) \in R^2$, from (5) we can obtain

$$QNu = (QNu_1, QNu_2).$$

where

$$QNu_1 = \bar{r}_1 - \bar{a}_{11}\exp\{u_1\} - \bar{a}_{12}\exp\{u_2\},$$
$$QNu_2 = -\bar{r}_2 + \bar{a}_{21}\exp\{u_1\} - \bar{a}_{22}\exp\{u_2\}.$$

We consider the following system of algebraic equations

$$\begin{aligned}
&\bar{r}_1 - \bar{a}_{11}v_1 - \bar{a}_{12}v_2 = 0, \\
&-\bar{r}_2 + \bar{a}_{21}v_1 - \bar{a}_{22}v_2 = 0.
\end{aligned}$$

By direct calculation we can get

$$v_1^* = \frac{\bar{r}_1\bar{a}_{22} + \bar{r}_2\bar{a}_{12}}{\bar{a}_{11}\bar{a}_{22} + \bar{a}_{12}\bar{a}_{21}},$$
$$v_2^* = \frac{\bar{r}_1\bar{a}_{21} - \bar{r}_2\bar{a}_{11}}{\bar{a}_{11}\bar{a}_{22} + \bar{a}_{12}\bar{a}_{21}} > \frac{\bar{a}_{21}[\bar{r}_1 - (\frac{a_{11}}{a_{21}})^M \bar{r}_2]}{\bar{a}_{11}\bar{a}_{22} + \bar{a}_{12}\bar{a}_{21}}.$$

From the assumption of Theorem 2, the system of algebraic equations has a unique positive solution $v^* = (v_1^*, v_2^*)$. Hence, the equation $QNu = 0$ has a unique solution $u^* = (u_1^*, u_2^*) = (\ln v_1^*, \ln v_2^*, \ln) \in R^2$.

Choosing constant $H > 0$ large enough such that $|u_1^*| + |u_2^*| < H$ and $H > H_1 + H_2$, we define a bounded open set $\Omega \subset X$ as follows

$$\Omega = \{u \in X : \|u\| < H\}.$$

It is clear that Ω satisfies conditions (a) and (b) of Lemma 3. On the other hand, by directly calculating we can obtain

$$\begin{aligned}
&\deg\{JQN, \Omega \cap \operatorname{Ker} L, (0,0)\} \\
&\quad = \operatorname{sgn}\begin{vmatrix} -\bar{a}_{11}K_1 - \bar{a}_{12}K_2 \\ \bar{a}_{21}K_1 - \bar{a}_{22}K_2 \end{vmatrix},
\end{aligned}$$

where $K_i = \exp\{u_i\}(i = 1, 2)$.

From the assumption of Theorem 2, we have

$$\begin{vmatrix} -\bar{a}_{11}K_1 - \bar{a}_{12}K_2 \\ \bar{a}_{21}K_1 - \bar{a}_{22}K_2 \end{vmatrix} \ne 0.$$

From this, we finally have

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, (0,0)\} \ne 0.$$

This shows that Ω satisfies condition (c) of Lemma 3. Therefore, system (4) has a ω-periodic solution $u^*(t) = (u_1^*(t), u_2^*(t)) \in \bar{\Omega}$. Hence, system (2) has a positive ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))$. □

Theorem 3 Suppose that assumptions of Theorem 2 hold. Further suppose that the following (H_2) holds.

(H_2) There exists a constant $\mu_i > 0$ $(i = 1, 2)$ such that

$$\liminf_{t\to\infty} A_i(t) > 0, \quad i = 1, 2,$$

where

$$\begin{aligned}A_1(t) &= \mu_1 a_{11}(t) - \mu_1 \int_{t-\tau_1}^{t} a_{11}(u+\tau_1)du[r_1(t)\\&\quad + (a_{11}(t) + a_{12}(t))M]\\&\quad - M\sum_{j=1}^{2} \mu_j\tau_j a_{jj}^M a_{j1}(t+\tau_j) - \mu_2 a_{21}(t+\tau_2),\\A_2(t) &= \mu_2 a_{22}(t) - \mu_2 \int_{t-\tau_2}^{t} a_{22}(u+\tau_2)du[r_2(t)\\&\quad + (a_{21}(t) + a_{22}(t))M]\\&\quad - M\sum_{j=1}^{2} \mu_j\tau_j a_{jj}^M a_{j2}(t+\tau_j) - \mu_1 a_{12}(t+\tau_1),\end{aligned}$$

where $M = \max\{M_1, M_2\}$ and M_1, M_2 are defined in Theorem 1. Then system (2) has a positive periodic solution which is globally attractive.

Proof From Theorem 2, we can obtain that system (2) has a positive periodic solution.

Let $x(t) = (x_1(t), x_2(t))$ be a positive periodic solution of system (2) and $(y_1(t), y_2(t))$ be a any positive solution of system (2). From Theorem 1, choose positive constants $m_i > 0$, $M_i > 0$ such that

$$m \le x_i(t) \le M, \quad i = 1, 2, \tag{19}$$

where $M = \max\{M_1, M_2\}$ and $m = \min\{m_1, m_2\}$ for all $t \ge T$. Let

$$W_1(t) = \sum_{i=1}^{2} \mu_i |\ln x_i(t) - \ln y_i(t)|.$$

Calculating the upper right derivation of $W_1(t)$ along system (2) for all $t \ge T$, we have

$$\begin{aligned}D^+W_1(t) &= \mu_1 \text{ sign } (x_1(t) - y_1(t))[-a_{11}(t)(x_1(t-\tau_1)\\&\quad - y_1(t-\tau_1)) - a_{12}(t)(x_2(t-\tau_1) - y_2(t-\tau_1))]\\&\quad + \mu_2 \text{ sign } (x_2(t) - y_2(t))\\&\quad \times [a_{21}(t)(x_1(t-\tau_2) - y_1(t-\tau_2))\\&\quad - a_{22}(t)(x_2(t-\tau_2) - y_2(t-\tau_2))]\\&= \mu_1 \text{ sign } (x_1(t) - y_1(t))[-a_{11}(t)(x_1(t) - y_1(t))\\&\quad - a_{12}(t)(x_2(t-\tau_1) - y_2(t-\tau_1))\\&\quad + a_{11}(t)\int_{t-\tau_1}^{t} (\dot{x}_1(u) - \dot{y}_1(u))du]\\&\quad + \mu_2 \text{ sign } (x_2(t) - y_2(t))[-a_{22}(t)(x_2(t) - y_2(t))\\&\quad + a_{21}(t)(x_1(t-\tau_2) - y_1(t-\tau_2))\\&\quad + a_{22}(t)\int_{t-\tau_2}^{t} (\dot{x}_2(u) - \dot{y}_2(u))du]\\&= \mu_1 \text{ sign } (x_1(t) - y_1(t))[-a_{11}(t)(x_1(t) - y_1(t))\\&\quad - a_{12}(t)(x_2(t-\tau_1) - y_2(t-\tau_1))\\&\quad + a_{11}(t)\int_{t-\tau_1}^{t} \big(x_1(u)[r_1(u) - a_{11}(u)x_1(u-\tau_1)\\&\quad - a_{12}(u)x_2(u-\tau_1)] - y_1(u)[r_1(u)\\&\quad - a_{11}(u)y_1(u-\tau_1) - a_{12}(u)y_2(u-\tau_1)]\big)du]\\&\quad + \mu_2 \text{ sign } (x_2(t) - y_2(t))[-a_{22}(t)(x_2(t) - y_2(t))\\&\quad + a_{21}(t)(x_1(t-\tau_2) - y_1(t-\tau_2))\\&\quad + a_{22}(t)\int_{t-\tau_2}^{t} \big(x_2(u)[r_2(u) + a_{21}(u)x_1(u-\tau_2)\\&\quad - a_{22}(u)x_2(u-\tau_2)] - y_2(u)[r_2(u)\\&\quad + a_{21}(u)y_1(u-\tau_2) - a_{22}(u)y_2(u-\tau_2)]\big)du]\\&= \mu_1 \text{ sign } (x_1(t) - y_1(t))[-a_{11}(t)(x_1(t) - y_1(t))\\&\quad - a_{12}(t)(x_2(t-\tau_1) - y_2(t-\tau_1))\\&\quad + a_{11}(t)\int_{t-\tau_1}^{t} \big((x_1(u) - y_1(u))[r_1(u) - a_{11}(u)y_1(u-\tau_1)\\&\quad - a_{12}(u)y_2(u-\tau_1)]\\&\quad + x_1(u)[-a_{11}(u)(x_1(u-\tau_1) - y_1(u-\tau_1))\\&\quad - a_{12}(u)(x_2(u-\tau_1) - y_2(u-\tau_1))]\big)du]\\&\quad + \mu_2 \text{ sign } (x_2(t) - y_2(t))[-a_{22}(t)(x_2(t) - y_2(t))\\&\quad + a_{21}(t)(x_1(t-\tau_2) - y_1(t-\tau_2))\\&\quad + a_{22}(t)\int_{t-\tau_2}^{t} \big((x_2(u) - y_2(u))[r_2(u) + a_{21}(u)y_1(u-\tau_2)\\&\quad - a_{22}(u)y_2(u-\tau_2)] + x_2(u)[a_{21}(u)(x_1(u-\tau_2)\\&\quad - y_1(u-\tau_2)) - a_{22}(u)(x_2(u-\tau_2) - y_2(u-\tau_2))]\big)du]\\&\le -\sum_{i=1}^{2} \mu_i a_{ii}(t)|x_i(t) - y_i(t)| + \mu_1 a_{12}(t)|x_2(t-\tau_1) - y_2(t-\tau_1)|\\&\quad + \mu_2 a_{21}(t)|x_1(t-\tau_2) - y_1(t-\tau_2)|\\&\quad + \mu_1 a_{11}(t)\int_{t-\tau_1}^{t} \big(|x_1(u) - y_1(u)|\big[r_1(u)\\&\quad + a_{11}(u)y_1(u-\tau_1) + a_{12}(u)y_2(u-\tau_1)] + x_1(u)\\&\quad [a_{11}(u)|x_1(u-\tau_1) - y_1(u-\tau_1)|\\&\quad + a_{12}(u)|x_2(u-\tau_1) - y_2(u-\tau_1)|]\big)du\\&\quad + \mu_2 a_{22}(t)\int_{t-\tau_2}^{t} \big(|x_2(u) - y_2(u)|\big[r_2(u) + a_{21}(u)y_1(u-\tau_2)\\&\quad + a_{22}(u)y_2(u-\tau_2)] + x_2(u)\big[a_{21}(u)|x_1(u-\tau_2)\\&\quad - y_1(u-\tau_2)| + a_{22}(u)|x_2(u-\tau_2) - y_2(u-\tau_2)|]\big)du.\end{aligned} \tag{20}$$

Define

$$W_2(t) = \mu_1 V_1(t) + \mu_2 V_2(t),$$

where

$$V_1(t) = \int_{t-\tau_1}^{t}\int_u^t a_{11}(u+\tau_1)\big(\big[r_1(s) + a_{11}(s)y_1(s-\tau_1) + a_{12}(s)y_2(s-\tau_1)\big]|x_1(s) - y_1(s)| + x_1(s)\big[a_{11}(s)|x_1(s-\tau_1) - y_1(s-\tau_1)| + a_{12}(s)|x_2(s-\tau_1) - y_2(s-\tau_1)|\big)dsdu,$$

$$V_2(t) = \int_{t-\tau_2}^{t}\int_u^t a_{22}(u+\tau_2)\big(\big[r_2(s) + a_{21}(s)y_1(s-\tau_2) + a_{22}(s)y_2(s-\tau_2)\big]|x_2(s) - y_2(s)| + x_2(s)\big[a_{21}(s)|x_1(s-\tau_2) - y_1(s-\tau_2)| + a_{22}(s)|x_2(s-\tau_2) - y_2(s-\tau_2)|\big)dsdu,$$

Calculating the upper right derivative, from (20) we have

$$\begin{aligned}\sum_{i=1}^{2} D^+W_i(t) \le & -\sum_{i=1}^{2}\mu_i a_{ii}(t)|x_i(t) - y_i(t)| \\ & + \mu_1 a_{12}(t)|x_2(t-\tau_1) - y_2(t-\tau_1)| \\ & + \mu_2 a_{21}(t)|x_1(t-\tau_2) - y_1(t-\tau_2)| \\ & + \mu_1\int_{t-\tau_1}^{t} a_{11}(u+\tau_1)du\big[r_1(t) \\ & + (a_{11}(t) + a_{12}(t))M\big]|x_1(t) - y_1(t)| \\ & + \mu_1\tau_1 a_{11}^M M\sum_{i=1}^{2} a_{1i}(t) \\ & \times |x_i(t-\tau_1) - y_i(t-\tau_1)| \\ & + \mu_2\int_{t-\tau_2}^{t} a_{22}(u+\tau_2)du\big[r_2(t) \\ & + (a_{21}(t) + a_{22}(t))M\big]|x_2(t) - y_2(t)| \\ & + \mu_2\tau_2 a_{22}^M M\sum_{i=1}^{2} a_{2i}(t) \\ & \times \sum_{i=1}^{2} a_{2i}(t)|x_i(t-\tau_2) - y_i(t-\tau_2)|. \end{aligned} \qquad (21)$$

Define

$$W_3(t) = \mu_1 V_3(t) + \mu_2 V_4(t),$$

where

$$V_3(t) = \mu_1\tau_1 a_{11}^M M\sum_{i=1}^{2}\int_{t-\tau_1}^{t} a_{1i}(u+\tau_1)|x_i(u) - y_i(u)|du + \int_{t-\tau_1}^{t} a_{12}(u+\tau_1)|x_2(u) - y_2(u)|du,$$

$$V_4(t) = \mu_2\tau_2 a_{22}^M M\sum_{i=1}^{2}\int_{t-\tau_1}^{t} a_{2i}(u+\tau_2)|x_i(u) - y_i(u)|du + \int_{t-\tau_2}^{t} a_{21}(u+\tau_2)|x_1(u) - y_1(u)|du.$$

Further, we define a Liapunov function as follows

$$V(t) = \sum_{i=1}^{3} W_i(t).$$

Calculating the upper right derivation of $V(t)$, from (20) and (21) we finally can obtain for all $t \ge T$

$$D^+V(t) \le -\sum_{i=1}^{2} A_i(t)|x_i(t) - y_i(t)|. \qquad (22)$$

From assumption (H_2), there exists a constant $\alpha > 0$ and $T^* \ge T$ such that for all $t \ge T^*$ we have

$$A_i(t) \ge \alpha > 0, \quad i = 1, 2. \qquad (23)$$

Integrating from T^* to t on both sides of (22) and by (23) produces

$$V(t) + \alpha\int_{T^*}^{t}\left(\sum_{i=1}^{2}|x_i(s) - y_i(s)|\right)ds \le V(T^*), \qquad (24)$$

then

$$\int_0^t \big(\sum_{i=1}^{2}|x_i(s) - y_i(s)|\big)ds \le \frac{V(T^*)}{\alpha}, \quad t \ge T^*. \qquad (25)$$

By the definition of $V(t)$ and (24) we have

$$\sum_{i=1}^{2}\mu_i|\ln x_i(t) - \ln y_i(t)| \le V(t) \le V(T^*), \quad t \ge 0. \qquad (26)$$

Therefore, for $i = 1, 2$ we have

$$\mu_i|\ln x_i(t) - \ln y_i(t)| \le V(T^*), \quad t \ge 0. \qquad (27)$$

which, together with (19), lead to

$$m_i\exp\{-V(T^*)/\mu_i\} \le y_i(t) \le M_i\exp\{V(T^*)/\mu_i\}, \quad i = 1, 2. \qquad (28)$$

From the boundedness of $x_i(t)$ and (25), it follows that $y_i(t)(i = 1, 2)$ are bounded for $t \ge 0$. From the boundedness of $x_i(t)$ and $y_i(t)$ we know that the derivatives $\dot{x}_i(t)$ and $\dot{y}_i(t)$ are bounded. Furthermore, we can obtain that $x_i(t) - y_i(t)$ and their derivatives remain bounded on $[0, +\infty)$. Therefore $\sum_{i=1}^{2}|x_i(t) - y_i(t)|$ is uniformly continuous on $[0, +\infty)$. By Barbalat's theorem it follows that:

$$\lim_{t\to+\infty}\sum_{i=1}^{2}|x_i(t) - y_i(t)| = 0.$$

Therefore,

$$\lim_{t\to+\infty}(x_i(t) - y_i(t)) = 0, \quad i = 1, 2.$$

This completes the proof of Theorem 3. □

From the global attractivity of bounded positive solutions, we have the following result.

Corollary 1 Suppose that the conditions of Theorem 3 hold, then system (2) is permanent.

As a direct corollary of Lemma 2, we have

Corollary 2 Suppose that $a_{21}^M M_1 < r_2^L$, then the predator species of system (2) goes to extinction.

Application

In this section, we will apply the results in Sect. 3 to the following predator–prey system with pure delays

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\big[r_1(t) - a_{11}(t)x_1(t-\tau_1) - a_{12}(t)x_2(t-\tau_1)\big],\\ \dot{x}_2(t) &= x_2(t)\big[r_2(t) + a_{21}(t)x_1(t-\tau_2) - a_{22}(t)x_2(t-\tau_2)\big].\end{aligned} \tag{29}$$

Corollary 3 Suppose that assumption (**H1**) holds, then there exist positive constants $N_i (i = 1, 2)$ such that

$$x_i(t) \le N_i,$$

for any positive solution $x_i(t)$ of system (29).

Corollary 4 Suppose that assumption (**H1**) holds and $\bar{r}_1 - (\frac{a_{12}}{a_{22}})^M \bar{r}_2 > 0$. Then system (29) has at least one positive $\omega-$ periodic solution.

Corollary 5 Suppose that assumptions of Corollary 4 hold. Further suppose that the following (H_3) holds.

(H_3) There exists a constant $v_i > 0$ $(i = 1, 2)$ such that

$$\liminf_{t\to\infty} B_i(t) > 0, \quad i = 1, 2,$$

where

$$\begin{aligned}B_1(t) = \quad & v_1 a_{11}(t) - v_1 \int_{t-\tau_1}^{t} a_{11}(u+\tau_1)du[r_1(t) + (a_{11}(t) + a_{12}(t))N]\\ & - N\sum_{j=1}^{2} v_j \tau_j a_{jj}^M a_{j1}(t+\tau_j) - v_2 a_{21}(t+\tau_2),\\ B_2(t) = \quad & v_2 a_{22}(t) - v_2 \int_{t-\tau_2}^{t} a_{22}(u+\tau_2)du[r_2(t) + (a_{21}(t) + a_{22}(t))N]\\ & - N\sum_{j=1}^{2} v_j \tau_j a_{jj}^M a_{j2}(t+\tau_j) - v_1 a_{12}(t+\tau_1),\end{aligned}$$

where $N = \max\{N_1, N_2\}$. Then system (29) has a positive periodic solution which is globally attractive.

Corollary 6 Suppose that the conditions of Theorem 3 hold, then system (29) is permanent.

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